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Engineering Notes

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Strong Hohmann Transfer Theorem

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Introduction

TWO lower bounds of velocity changes for orbital transfer from a circular orbit into a coplanar conical orbit (and vice versa) are found. As these values of two bounds are just the same as those used in the Hohmann transfer between two coplanar circular orbits, we call this result the strong Hohmann transfer theorem. This theorem also means that, for any given ΔV, the maximum radius change from a circular orbit can be attained by applying the ΔV tangentially. It seems that this is one of those facts everyone "knows"; however, we have not found any proof of this property so far.

One of the direct applications of the strong Hohmann transfer theorem is the demonstration of the optimality of the Hohmann transfer. Consequently, it is shown that the Hohmann transfer is optimal, compared with other two-impulse transfer, not only in total velocity change but also in each velocity change.

Strong Hohmann Transfer Theorem

Let us denote the gravitational constant of the central body by μ and assume that all transfer orbits are conic and coplanar with the orbit in which a spacecraft was originally located . At first we consider the case, we call it the outer case, where a spacecraft is transferred to an outside orbit.

Theorem 1 (outer case). When a spacecraft moving in a circular orbit of radius R_1 is transferred to a coplanar conic orbit whose apoapsis is not smaller than R_2 ($R_2 > R_1$), it must be taken a velocity change of, at least, $[\sqrt{2R_2/(R_1+R_2)} - 1]\sqrt{\mu/R_1}$ at the departure point.

Proof of Theorem 1. We assume that

$$a(1+e) \geq R_2 \tag{1}$$

where a, e, and p mean the major semiaxis, the eccentricity, and the semi-latus rectum of the conic orbit, respectively.

From the two-body problem, we get the relations[1]

$$p = (R_1 V_a \cos\phi)^2/\mu \tag{2}$$

$$p = a(1-e^2) \tag{3}$$

$$V_a^2 = \mu(2/R_1 - 1/a) \tag{4}$$

and

$$V_a^2 = V_1^2 + \delta V^2 + 2V_1\,\delta V\,\cos\psi \tag{5}$$

where δV is the velocity change at the departure point, ψ is the angle between δV and the spacecraft's circular velocity V_1 ($V_1 = \sqrt{\mu/R_1}$), V_a is the spacecraft's velocity just after the velocity change at the departure point, and ϕ is the angle between V_a and V_1 (see Fig. 1).

At first we assume that $a > R_2$. Then, from Eq. (4), we get $V_a^2 > \mu(2/R_1 - 1/R_2)$. Now, from the triangle inequality, we have $\delta V \geq V_a - V_1$, and this implies that

$$\delta V \geq V_a - V_1 > \sqrt{\mu(2/R_1 - 1/R_2)} - V_1 \geq \left(\sqrt{2R_2/(R_1+R_2)} - 1\right)\sqrt{\mu/R_1} \tag{6}$$

The last inequality holds good when $V_1 = \sqrt{\mu/R_1}$.

Next we assume that $a \leq R_2$. Then, using Eqs. (1) and (3), we have

$$R_2 - a \leq \sqrt{a^2 - ap} \tag{7}$$

where $R_2 - a \geq 0$, so, from the square of Eq. (7), the following inequality is derived:

$$a(2R_2 - p) \geq R_2^2 \tag{8}$$

Using Eqs. (2), (4), and (5) and the law of sines ($\sin\phi/\delta V = \sin\psi/V_a$), we can represent a and p as the functions of δV and ψ. And finally we can transform the inequality (8) into the form

$$\left\{1 + \left[R_1^2/\left(R_2^2 - R_1^2\right)\right]\sin^2\psi\right\}V_0^2 + 2\cos\psi\, V_0 \geq [(R_2 - R_1)/(R_2 + R_1)] \tag{9}$$

where $V_0 = \delta V/V_1$. Solving this inequality with respect to V_0, we can verify that

$$V_0 \geq (R_2 - R_1)/\left[\sqrt{R_2^2 + \left(R_2^2 + 2R_2R_1\right)\cos^2\psi} + (R_2 + R_1)\cos\psi\right]$$
$$\geq (R_2 - R_1)/\left[\sqrt{R_2^2 + \left(R_2^2 + 2R_2R_1\right)} + (R_2 + R_1)\right]$$
$$= \sqrt{2R_2/(R_1 + R_2)} - 1$$

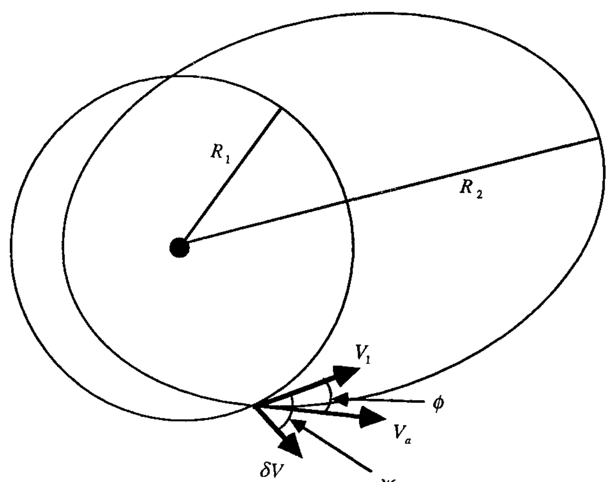

Fig. 1 Geometric relation among velocities of orbital transfer.

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and hence

$$\delta V \geq \left[\sqrt{2R_2/(R_1+R_2)} - 1\right]\sqrt{\mu/R_1} \tag{10}$$

The propriety of Theorem 1 is demonstrated by Eqs. (6) and (10).

By symmetry, this theorem can also be described in the following way:

Corollary 1. When a spacecraft moving in a conic orbit whose apoapsis is not smaller than R_2 is transferred to a coplanar circular orbit of radius $R_1 (R_1 < R_2)$, it must be taken a velocity change of at least $[\sqrt{2R_2/(R_1+R_2)} - 1]\sqrt{\mu/R_1}$ at the departure point.

Then, we consider the case, we call it the *inner case*, where a spacecraft is transferred to an inside orbit.

Theorem 2 (inner case). When a spacecraft moving in a circular orbit of radius R_2 is transferred to a coplanar conic orbit whose apoapsis is not larger than $R_1 (R_1 < R_2)$, it must be taken a velocity change of at least $[1 - \sqrt{2R_1/(R_1+R_2)}]\sqrt{\mu/R_2}$ at the departure point.

Proof of Theorem 2. We take the similar notations as those used in the proof of Theorem 1, and hence the V_1 and R_1 in Fig. 1 are replaced by V_2 and R_2. Now the requirement is that

$$a(1-e) \leq R_1 \tag{11}$$

From Eq. (3), we have

$$a - R_1 \leq \sqrt{a^2 - ap} \tag{12}$$

For the case of $a < R_1$, we can similarly derive that

$$\delta V \geq V_2 - V_a > V_2 - \sqrt{\mu(2/R_2 - 1/R_1)}$$

$$\geq \left[1 - \sqrt{2R_1/(R_1+R_2)}\right]\sqrt{\mu/R_2} \tag{13}$$

The last inequality holds good when $V_2 = \sqrt{\mu/R_2}$.

For the case of $a \geq R_1$, taking the square of Eq. (12), we have

$$a(2R_1 - p) \geq R_1^2 \tag{14}$$

And finally we can get

$$\left\{1 - \left[R_2^2/\left(R_2^2 - R_1^2\right)\right]\sin^2\psi\right\}V_0^2 + 2\cos\psi V_0$$

$$\leq -[(R_2 - R_1)/(R_2 + R_1)] \tag{15}$$

where $V_0 = \delta V/V_2$.

We must note here that, although inequality (15) is very similar to inequality (9) in expression, they are clearly different because the signs of inequality are just opposite in these two equations:

1) If $1 - [R_2^2/(R_2^2 - R_1^2)]\sin^2\psi \neq 0$, then we can have

$$V_0 \geq \left[(R_2 - R_1)/\left(R_1^2 - R_2^2\cos^2\psi\right)\right]$$

$$\times\left[\sqrt{(1+\cos^2\psi)R_1^2 + 2R_1R_2\cos^2\psi} + (R_1+R_2)\cos\psi\right]$$

$$= (R_2 - R_1)\Big/\left[\sqrt{(1+\cos^2\psi)R_1^2 + 2R_1R_2\cos^2\psi}\right.$$

$$\left. - (R_1+R_2)\cos\psi\right]$$

$$\geq (R_2 - R_1)\Big/\left[\sqrt{2R_1^2 + 2R_1R_2} + (R_1+R_2)\right] \tag{16}$$

2) If $1 - [R_2^2/(R_2^2 - R_1^2)]\sin^2\psi = 0$, then from Eq. (15) we can see that $\cos\psi = -R_1/R_2$, and

$$V_0 \geq (R_2 - R_1)R_2/[2(R_2+R_1)R_1] \tag{17}$$

so we only need to prove

$$(R_2 - R_1)R_2/[2(R_2+R_1)R_1] \geq 1 - \sqrt{2R_1/(R_1+R_2)} \tag{18}$$

This inequality is equivalent to

$$\sqrt{2R_1/(R_1+R_2)} \geq (2R_1/R_2) - 1 \tag{19}$$

and it obviously holds good when $2R_1 \leq R_2$. Taking a square of both sides of inequality (19) and assuming $2R_1 > R_2$, we can get the result that inequality (19) is equivalent to

$$(R_1 - R_2)\left(4R_1^2 + 4R_1R_2 - R_2^2\right) \leq 0 \tag{20}$$

This inequality holds good when $2R_1 > R_2$ and so the propriety of Theorem 2 is proved.

Also, Theorem 2 can be described as follows:

Corollary 2. When a spacecraft moving in a conic orbit whose periapsis is not larger than R_1 is transferred to a coplanar circular orbit of radius $R_2 (R_2 > R_1)$, it must be taken a velocity change of at least $[1 - \sqrt{2R_1/(R_1+R_2)}]\sqrt{\mu/R_2}$ at the departure point.

Proof of Global Optimality of Hohmann Transfer

We consider a two-impulse transfer between two coplanar circular orbits from an initial circular orbit of radius R_1 to a target orbit of radius R_2. Various kinds of proofs of the Hohmann transfer theorem have been published so far,[2–7] but only the optimality of the total amount of velocity changes has been considered. In our approach, however, two velocity changes are considered separately, and the least energy requirement for each velocity change is obtained.

Without loss of generality, we can assume that $R_1 < R_2$. The conic transfer orbit between these two circular orbits must meet the condition that its periapsis is not larger than R_1 and its apoapsis is not smaller than R_2. At first we consider the velocity change at the departure point, from the initial circular orbit to the conic orbit. We can see that this orbital change satisfies the condition of Theorem 1, and so there is a lower bound in the velocity change:

Velocity change for leaving

$$\geq \left[\sqrt{2R_2/(R_1+R_2)} - 1\right]\sqrt{\mu/R_1}$$

Then, we consider the velocity change at the destination point, from the conic transfer orbit to the circular target orbit. We can also see this orbital change satisfies the condition of Corollary 2, and so there is a lower bound in this velocity change:

Velocity change for entering

$$\geq \left[1 - \sqrt{2R_1/(R_1+R_2)}\right]\sqrt{\mu/R_2}$$

The total velocity change ΔV required for the transfer is the sum of the these two velocity changes. This means that for any conic transfer orbit,

$$\Delta V \geq \left[\sqrt{2R_2/(R_1+R_2)} - 1\right]\sqrt{\mu/R_1}$$

$$+\left[1 - \sqrt{2R_1/(R_1+R_2)}\right]\sqrt{\mu/R_2} \tag{21}$$

The right side of the inequality is the well-known velocity change of the Hohmann transfer.

Summary

We establish a theorem called the strong Hohmann transfer theorem, which gives the lower bounds of velocity changes in orbital transfer. This theorem also means that, for any given ΔV, the maximum radius change from a circular orbit can be attained by applying the ΔV tangentially. It seems that this is one of those facts everyone knows as the Hohmann transfer; however, we have not found any proof of this property so far. As shown in this Note, direct application of our theorem provides a unique proof of the Hohmann transfer theorem.

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Uniform Modal Damping of Rings by an Extended Node Control Theorem

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Introduction

IN principle, a control system can dampen any combination of the modes of a vibrating structure. In certain applications, one wants to preserve the uncontrolled mode shapes and resonant frequencies of the lower modes. When point control forces and sensors are used, the placement of the control forces and sensors on the structure plays an important role in achieving these objectives. To this end, the natural control theory provides a general strategy for designing a control system that preserves the resonant frequencies and mode shapes of the structure while uniformly damping the modes.[1–3] Natural control is an application of the independent modal space control method due to Meirovitch and Baruh.[4] Weaver and Silverberg[5] applied natural control to uniform beams and developed a control force and sensor placement strategy called node control. This strategy is based on the node control theorem, which states: "If a uniform beam with homogeneous boundary conditions and with the lowest N modes participating in the system response is subject to N direct state feedback control forces placed at the N nodes of the $(N+1)$th mode, then the control gains can be selected such that the following properties apply to the controlled system: 1) frequency invariance; 2) mode invariance; 3) uniform damping."[5]

This Note extends the node control theorem for uniform beams to uniform rings. In contrast with uniform beams, uniform rings possess no boundary conditions and they are degenerate in the sense that, for each natural frequency, there are two orthogonal mode shapes, i.e., a sine mode and a cosine mode. In this Note we shall show that $2M+2$ control forces evenly spaced around the ring can uniformly dampen the lowest $2M+1$ modes and preserve the associated natural frequencies and mode shapes.

Elastic Ring Equations

The equations of motion for a uniform ring can be written as[5]

$$\begin{aligned} &\rho A\ddot{v} - \left(\frac{EA}{R^2} + \frac{EI}{R^2}\right)\frac{\partial^2 v}{\partial\theta^2} - \frac{EA}{R^2}\frac{\partial w}{\partial\theta} + \frac{EI}{R^4}\frac{\partial^3 w}{\partial\theta^3} = 0 \\ &\rho A\ddot{w} - \frac{EA}{R^2}\frac{\partial v}{\partial\theta} - \frac{EI}{R^4}\frac{\partial^3 v}{\partial\theta^3} + \frac{EA}{R^2}w + \frac{EI}{R^4}\frac{\partial^4 w}{\partial\theta^4} = r + u \end{aligned} \tag{1}$$

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where ρ is the mass density of the ring, E is Young's modulus, A is the cross-sectional area, R is the mean radius, w is the normal displacement of the ring, and v is the in-plane motion. Both w and v are functions of location θ and time t. The excitation in the normal direction is r, and u is the normal control action. Both r and u are functions of θ and t. It is assumed that there is no in-plane external force so that the in-plane inertial term is negligible. This assumption simplifies the analysis since the first equation of (1) becomes time independent. However, the assumption has little effect on the natural frequencies of the ring.

Since the ring is uniform and continuous, the above equations are subject to no boundary conditions. Instead, we impose the conditions that w, v, r, and u be periodic functions of θ. The modal expansions of these quantities are given by

$$\begin{aligned} w(\theta,t) &= \sum_{n=-M}^{M} w_n(t)e^{jn\theta}, \qquad v(\theta,t) = \sum_{n=-M}^{M} v_n(t)e^{jn\theta} \\ u(\theta,t) &= \sum_{n=-M}^{M} u_n(t)e^{jn\theta}, \qquad r(\theta,t) = \sum_{n=-M}^{M} r_n(t)e^{jn\theta} \end{aligned} \tag{2}$$

where $j^2=-1$, w_n and v_n are modal displacements, r_n is a modal excitation, and u_n is a modal control force. The expansions are truncated at $n=\pm M$. Note that in order for the expansions of Eq. (3) to be real valued, the coefficients corresponding to n must be the complex conjugates of those corresponding to $-n$. Each term in the expansion of $w(\theta,t)$ represents a wave traveling around the ring and will be loosely referred to as a complex mode, or simply a mode of the ring. The sum of the $+n$th mode and the $-n$th mode results in a real deformation pattern consisting of a sine mode and a cosine mode. The number of nodes of the real deformation pattern is equal to $2n$. The nodes are evenly spaced along the ring and may move around the ring as the phase of the sine and cosine modes changes.

Substituting the expansions of Eqs. (2) into the equations of motion (1) leads to a set of decoupled equations for the coefficients w_n and v_n. The first equation of (1) yields an algebraic relationship between v_n and w_n, which allows the elimination of v_n from the second equation of (1), resulting in a set of second-order ordinary differential equations:

$$\ddot{w}_n + \omega_n^2 w_n = \hat{r}_n + \hat{u}_n, \qquad -M \le n \le M \tag{3}$$

where $\hat{r}_n = r_n/\rho A$ and $\hat{u}_n = u_n/\rho A$ are the normalized excitation and control modal forces and ω_n is the open-loop natural frequency of the nth mode given by

$$\omega_n^2 = \begin{cases} \dfrac{E}{\rho R^2}, & n=0 \\ \dfrac{1}{\rho A}\left(\dfrac{EA}{R^2} + \dfrac{EI}{R^4}n^4 - \dfrac{[EA/R^2 + (EI/R^4)n^2]^2}{EA/R^2 + EI/R^4}\right) & n \ne 0 \end{cases} \tag{4}$$

Control System

Consider N discrete point control forces expressed as

$$u(\theta,t) = \sum_{i=1}^{N} f_i(t)\,\delta(\theta-\theta_i) = \sum_{i=1}^{N}[h_i\dot{w}(\theta_i,t) + g_i w(\theta_i,t)]\,\delta(\theta-\theta_i) \tag{5}$$

where h_i is the velocity feedback gain and g_i is the position feedback gain of the ith control force $f_i(t)$ located at θ_i. Then,

$$\hat{u}_n = \sum_{m=-M}^{M} H_{nm}\dot{w}_m + \sum_{m=-M}^{M} G_{nm}w_m \tag{6}$$